B.Sc. Ist Part (Hons) Ist Paper

Trigonometry (contd.)

Summation of series

1 Find the sum of the series

$3\sin\alpha + 5\sin 2\alpha + 7\sin 3\alpha + \cdots$ to n-terms.

Soln

Let $S = 3\sin\alpha + 5\sin 2\alpha + 7\sin 3\alpha + \cdots$ to n-terms

Now, let $C = 3\cos\alpha + 5\cos 2\alpha + 7\cos 3\alpha + \cdots$ to n-terms

$\therefore\ C + iS = 3(\cos\alpha + i\sin\alpha) + 5(\cos 2\alpha + i\sin 2\alpha)$
$+ 7(\cos 3\alpha + i\sin 3\alpha) + \cdots$ to n-terms.

$\Rightarrow C + iS = 3e^{i\alpha} + 5e^{2i\alpha} + 7e^{3i\alpha} + \cdots$ to n-terms.

n^{th} term of $3, 5, 7, \cdots$ (which are in AP) $= 3 + (n-1)\times 2$
$= 2n+1$.

n^{th} term of $e^{i\alpha}, e^{2i\alpha}, e^{3i\alpha}, \cdots$ which are in GP
$= e^{i\alpha}\cdot(e^{i\alpha})^{n-1} = e^{ni\alpha}$.

$\Rightarrow C + iS = 3e^{i\alpha} + 5e^{2i\alpha} + 7e^{3i\alpha} + \cdots + (2n+1)e^{ni\alpha}$ —(1)

Multiplying both sides by $e^{i\alpha}$, we get

$\Rightarrow (C+iS)e^{i\alpha} = 3e^{2i\alpha} + 5e^{3i\alpha} + 7e^{4i\alpha} + \cdots + (2n+1)e^{(n+1)i\alpha}$ —(2)

Subtracting (2) from (1), we get

$$\therefore (C+iS)(1-e^{i\alpha}) = 3e^{i\alpha} + 2e^{2i\alpha} + 2e^{3i\alpha} + \cdots + 2e^{ni\alpha} - (2n+1)e^{(n+1)i\alpha}$$

$$\Rightarrow (C+iS)(1-e^{i\alpha}) = 3e^{i\alpha} + 2(e^{2i\alpha} + e^{3i\alpha} + \cdots + e^{ni\alpha}) - (2n+1)e^{(n+1)i\alpha}$$

$$\Rightarrow (C+iS)(1-e^{i\alpha}) = e^{i\alpha} + 2(e^{i\alpha} + e^{2i\alpha} + e^{3i\alpha} + \cdots + e^{ni\alpha}) - (2n+1)e^{(n+1)i\alpha}$$

$$= e^{i\alpha} + 2\left[\frac{1-(e^{i\alpha})^n}{1-e^{i\alpha}}\right] e^{i\alpha} - (2n+1)e^{(n+1)i\alpha}$$

$$= e^{i\alpha} + 2 \cdot \frac{1-e^{ni\alpha}}{(1-e^{i\alpha})e^{-i\alpha}} - (2n+1)e^{(n+1)i\alpha}$$

$$= e^{i\alpha} - \frac{2(1-e^{ni\alpha})}{(-e^{-i\alpha}+1)} - (2n+1)e^{(n+1)i\alpha}$$

$$\Rightarrow C+iS = \frac{1}{1-e^{i\alpha}}\left[e^{i\alpha} - \frac{2(1-e^{ni\alpha})}{1-e^{-i\alpha}} - (2n+1)e^{(n+1)i\alpha}\right]$$

$$\Rightarrow C+iS = \frac{1}{1-e^{i\alpha}}\left[e^{i\alpha} - (2n+1)e^{(n+1)i\alpha} - \frac{2(1-e^{ni\alpha})}{1-e^{-i\alpha}}\right]$$

$$\Rightarrow C+iS = \frac{e^{i\alpha} - (2n+1)e^{(n+1)i\alpha}}{1-e^{i\alpha}} - \frac{2(1-e^{ni\alpha})}{(1-e^{-i\alpha})(1-e^{i\alpha})}$$

$$\Rightarrow c+is = \left[\frac{e^{i\alpha}-(2n+1)e^{(n+1)i\alpha}}{1-e^{i\alpha}}\right]\times\frac{1-e^{-i\alpha}}{1-e^{-i\alpha}}$$

$$-\frac{2\left(1-e^{ni\alpha}\right)}{\left[1-\left(e^{i\alpha}+e^{-i\alpha}\right)+e^{i\alpha}.e^{-i\alpha}\right]}$$

$$\Rightarrow c+is = \frac{\left[e^{i\alpha}-(2n+1)e^{(n+1)i\alpha}\right]\left(1-e^{-i\alpha}\right)}{1-\left(e^{i\alpha}+e^{-i\alpha}\right)+e^{i\alpha}\cdot e^{-i\alpha}}$$

$$-\frac{2\left(1-e^{ni\alpha}\right)}{1-2\cos\alpha+1}$$

$$\Rightarrow c+is = \frac{e^{i\alpha}-1-(2n+1)e^{(n+1)i\alpha}+(2n+1)e^{ni\alpha}}{1-2\cos\alpha+1}$$

$$-\frac{2\left(1-e^{ni\alpha}\right)}{2\left(1-\cos\alpha\right)}.$$

$$\Rightarrow c+is = \frac{e^{i\alpha}-1-(2n+1)e^{(n+1)i\alpha}+(2n+1)e^{ni\alpha}-2+2e^{ni\alpha}}{2\left(1-\cos\alpha\right)}$$

$$\Rightarrow c+is = \frac{e^{i\alpha}-3-(2n+1)e^{(n+1)i\alpha}+(2n+3)e^{ni\alpha}}{2\left(1-\cos\alpha\right)}$$

~~⇒ c+is~~ Now expanding $e^{i\theta}=\cos\theta+i\sin\theta$,
we get

$$C + iS = \frac{(\cos\alpha + i\sin\alpha) - 3 - (2n+1)\left[\cos(n+1)\alpha + i\sin(n+1)\alpha\right] + (2n+3)(\cos n\alpha + i\sin n\alpha}{2(1-\cos\alpha)}$$

Equating imaginary parts, we have

$$\Rightarrow \quad S = \frac{\sin\alpha - (2n+1)\sin(n+1)\alpha + (2n+3)\sin n\alpha}{2(1-\cos\alpha)}$$

Ans